Lecture 17
14.7 Extreme values and saddle points

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March 1, 2019

Things to note

Upcoming dates:
Monday: Quiz 8 and WF drop date (see grade calculation sheet on Blackboard)
Wednesday, March 6: Review
Friday, March 8: Exam 2

Last class

Then the equation of the plane tangent to $f(x, y)$ at $(a, b, f(a, b))$ is

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) - (z - f(a, b)) = 0.$$

Solving for z, we have

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

Tangent plane example

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Example

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We have $\nabla f = \langle \cos(y) - ye^x, -x\sin(y) - e^x \rangle$ and thus the equation is

$$(x - \ln(2)) + (-2)(y - 0) - (z - \ln(2)) = 0$$

or

$$z = x - 2y.$$

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Let (a, b) be in the domain of $f(x, y)$. We say

1. *$f(a, b)$ is a local maximum of f if $f(a, b) \geq f(x, y)$ for all points (x, y) in the domain of f near (a, b).*

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Example

The function $f(x, y) = x^2 + y^2$ has a local min at $(0, 0)$.

First derivative test

Theorem

If $f(x, y)$ has a local min or max at (a, b) and $f_x(a, b)$, $f_y(a, b)$ are defined, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Another way to say this is $\nabla f(a, b) = \vec{\mathbf{0}}$.

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Definition

A point (a, b) in the domain of f is a <u>*critical point of f*</u> *if $f_x(a, b) = 0 = f_y(a, b)$ or if $f_x(a, b)$ or $f_y(a, b)$ is undefined.*

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$$\nabla f = \langle 2x, 2y - 4 \rangle.$$

This gives the equations $2x = 0$ and $2y - 4 = 0$ or $x = 0, y = 2$. Thus there MIGHT be an extreme value of f at (0,2). At this point, $f(0, 2) = 0 + 2^2 - 4(2) + 9 = 5$. Can we tell if this is a local max or a local min?

$$f(x, y) = x^2 + (y - 2)^2 + 5 \geq 0 + 0 + 5 = 5$$

So $(0, 2)$ is a local min since the function is always higher than height 5.

Another example

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We have $f_x = -2x$ and $f_y = 2y$. Setting these equal to 0 gives $x = 0$ and $y = 0$. So $(0,0)$ is a critical point and $f(0,0) = 0$ MIGHT be an extreme value of f. Can we tell if this height is a local max or a local min?

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Choose a path where f looks like a max at (0,0):

$$\text{along } y = 0 : f(x,0) \leq 0 = f(0,0)$$

Choose a path where f looks like a min at $(0,0)$:

$$\text{along } x = 0 : f(0,y) = y^2 \geq 0 = f(0,0)$$

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We have $f_x = -2x$ and $f_y = 2y$. Setting these equal to 0 gives $x = 0$ and $y = 0$. So $(0, 0)$ is a critical point and $f(0, 0) = 0$ MIGHT be an extreme value of f. Can we tell if this height is a local max or a local min?

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Choose a path where f looks like a min at $(0, 0)$:

$$\text{along } x = 0 : f(0, y) = y^2 \geq 0 = f(0, 0)$$

So $(0, 0)$ is neither a local min nor a local max.

Saddle points

Definition

$f(x, y)$ has a saddle point at a critical point (a, b) if (a, b) isn't a local max and (a, b) isn't a local min.
The point $(a, b, f(a, b))$ is called a saddle point.

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In general, critical values don't come in nice cookie-cutter ways that allow us to easily figure out whether the points are local extrema or saddle points. But there is a test we can use to determine what MOST critical values are.